

An application of the light-front coupled-cluster method to the nonperturbative solution of QED

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Abstract

The new light-front coupled-cluster (LFCC) method for the nonperturbative solution of Hamiltonian eigenvalue problems is described and then illustrated in an application to quantum electrodynamics. The method eliminates any necessity for a Fock-space truncation and thereby avoids complications associated with such a truncation. An LFCC calculation of the electron's anomalous magnetic moment is formulated for a truncation that, for simplicity, excludes positrons but keeps arbitrarily many photons.

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I. INTRODUCTION

In order to have a better understanding of hadronic physics within quantum chromodynamics, it is imperative to have methods for the nonperturbative solution of the theory. Lattice gauge theory [1] has been quite successful in this regard. The use of Dyson–Schwinger equations [2] has also born fruit. Both, however, are somewhat indirect, in that they do not provide Minkowski-space wave functions from which one could compute observables directly. An alternative that does deal directly with such wave functions is the light-front Hamiltonian method [3]. There wave functions appear as coefficients in Fock-state expansions of the eigenstates.

The light-front Hamiltonian approach now has long history [3], beginning with Dirac’s initial formulation of the coordinate choice [4]. A key ingredient in practical calculations has been the truncation of Fock space. This yields a finite set of equations for a finite set of wave functions, which are usually solved numerically.

Unfortunately, such truncations cause several difficulties, including uncanceled divergences [5], broken Ward identities [6], and loss of Lorentz covariance [7, 8] and gauge invariance [9]. These come from the nonperturbative analog of decomposing a Feynman diagram into time-ordered contributions and throwing away those contributions that involve more than some fixed number of intermediate particles. Some of these difficulties can be somewhat mitigated by use of a sector-dependent parameterization [10–12], but they are not eliminated and can, in fact, lead to ill-defined wave functions [13].

To avoid these difficulties, we have recently proposed a light-front coupled-cluster (LFCC) method [14]. Fock space is not truncated. Instead, an eigenstate is constructed from a valence state $|\phi\rangle$ by the action of an exponentiated operator T . With inclusion of a normalizing factor \sqrt{Z} , the eigenstate is written as $\sqrt{Z}e^T|\phi\rangle$. The problem is then to find $|\phi\rangle$ and T . To make the problem finite, T is truncated, but the exponential of T is not, so that $e^T|\phi\rangle$ includes contributions from all relevant Fock sectors. The method relies on light-front coordinates, to make Fock-state expansions well-defined and to facilitate separation of internal and external momenta, and on the mathematical techniques of the many-body coupled-cluster method [15, 16], hence the name.

The original coupled-cluster method is used primarily to solve many-body problems where the number of particles is large and unchanging. It was first developed by Coester and Kümmel [15] for applications to the many-body Schrödinger equation in nuclear physics. It was extended to many-electron problems in molecules by Čížek [17], which was eventually followed by extensive development in quantum chemistry [16]. The basic idea is to form an eigenstate as $e^T|\phi\rangle$, where $|\phi\rangle$ is a product of single-particle states, as in the Hartree–Fock approximation, and the terms in T annihilate states in $|\phi\rangle$ and create excited states, to build in correlations. A finite numerical calculation is then done by truncating T at some (small) number of excitations. There has also been a series of applications to field theory [18]. These rely on Fock-state expansions in equal time and necessarily focus on the structure of the vacuum. Particle states are built on this vacuum. There was some success in analyzing ϕ^4 theory in $1+1$ dimensions.

Our purpose here is to illustrate the LFCC method in a concrete calculation in a gauge theory. We consider the state of the dressed electron and its anomalous magnetic moment in an arbitrary covariant gauge. The theory is regulated by the inclusion of Pauli–Villars (PV) photons and electrons [5, 19]. We do truncate Fock space but only to exclude positrons; all possible single-electron, multi-photon states are retained. The exclusion of positrons is not

required by the method, but is done to simplify the illustration.

We begin in Sec. II with an extended description of the LFCC method [14]. The application to QED is developed in the following two sections; Sec. III focuses on the eigenvalue problem, and Sec. IV on the calculation of the anomalous moment. The details of light-front QED in an arbitrary covariant gauge [9] are discussed in Appendix A, including a new analysis of the gauge projection. Details of the construction of the effective LFCC Hamiltonian are given in Appendix B.

II. A LIGHT-FRONT COUPLED-CLUSTER METHOD

We use light-front coordinates [3, 4], where time evolves along $x^+ \equiv t + z$. The spatial coordinates are $\underline{x} = (x^-, \vec{x}_\perp)$, with $x^- \equiv t - z$ and $\vec{x}_\perp = (x, y)$. The light-front energy is $p^- \equiv E - p_z$ and momentum $\underline{p} = (p^+, \vec{p}_\perp)$, with $p^+ \equiv E + p_z$ and $\vec{p}_\perp = (p_x, p_y)$. The mass-shell condition $p^2 = m^2$ becomes $p^- = (m^2 + p_\perp^2)/p^+$. In a system with total momentum \underline{P} , a constituent with momentum \underline{p} is defined to have a longitudinal momentum fraction $y \equiv p^+/P^+$ and relative transverse momentum $\vec{k}_\perp = \vec{p}_\perp - y\vec{P}_\perp$. A creation operator $a^\dagger(\underline{p})$ for such a particle will be written as $a^\dagger(y, \vec{k}_\perp; \underline{P})$. Clearly, we have $a^\dagger(1, 0; \underline{P}) = a^\dagger(\underline{P})$.

Given a light-front Hamiltonian \mathcal{P}^- , we wish to solve the fundamental eigenvalue problem

$$\mathcal{P}^- |\psi(\underline{P})\rangle = \frac{M^2 + P_\perp^2}{P^+} |\psi(\underline{P})\rangle \quad (2.1)$$

for the eigenmass M and the eigenstate $|\psi(\underline{P})\rangle$, in a basis where the momentum \underline{P} is diagonal with eigenvalue \underline{P} . The normalization is chosen to be

$$\langle \psi(\underline{P}') | \psi(\underline{P}) \rangle = \delta(\underline{P}' - \underline{P}). \quad (2.2)$$

The eigenstate can be generically expanded in a Fock basis $\{|n; \underline{p}_i\rangle\}$ as

$$|\psi(\underline{P})\rangle = \sum_n \int \left(\prod_i dy_i d\vec{k}_{i\perp} \right) \delta(1 - \sum_i y_i) \delta(\sum_i \vec{k}_{i\perp}) \psi_n(y_i, \vec{k}_{i\perp}) |n; \underline{p}_i\rangle, \quad (2.3)$$

where n is the number of constituents and \underline{p}_i their light-front momenta. The eigenvalue problem then becomes a coupled set of integral equations for the wave functions ψ_n . In this form, the problem requires truncation to a finite set of wave functions, and the various complications follow, as discussed in the Introduction.

To avoid truncation of the Fock space, we construct the eigenstate as

$$|\psi(\underline{P})\rangle = \sqrt{Z} e^T |\phi(\underline{P})\rangle. \quad (2.4)$$

The valence state $|\phi(\underline{P})\rangle$ contains a small number of constituents and will be obtained from diagonalization of an effective Hamiltonian in the corresponding valence sector, where the small number of particles is fixed. The operator T is constructed to preserve all the quantum numbers of the valence state, including momentum \underline{P} and angular momentum projection J_z , and to always increase the number of constituents. The exponential of T then generates all the higher Fock states and their associated wave functions. The normalizing factor \sqrt{Z} is chosen to allow the valence state to have the same momentum normalization as the full state:

$$\langle \phi(\underline{P}') | \phi(\underline{P}) \rangle = \delta(\underline{P}' - \underline{P}). \quad (2.5)$$

The eigenvalue problem can now be written as

$$\overline{\mathcal{P}}^-|\phi(\underline{P})\rangle = \frac{M^2 + P_\perp^2}{P^+}|\phi(\underline{P})\rangle, \quad (2.6)$$

with $\overline{\mathcal{P}}^- \equiv e^{-T}\mathcal{P}^-e^T$ the effective Hamiltonian. By introducing a projection P_v onto the valence sector, defined as the sector of Fock space containing the Fock states found in the valence state, we separate the fundamental eigenvalue problem into a valence eigenvalue problem

$$P_v\overline{\mathcal{P}}^-|\phi(\underline{P})\rangle = \frac{M^2 + P_\perp^2}{P^+}|\phi(\underline{P})\rangle \quad (2.7)$$

to be solved for the valence state, and a set of auxiliary equations

$$(1 - P_v)\overline{\mathcal{P}}^-|\phi(\underline{P})\rangle = 0 \quad (2.8)$$

that determine the operator T . The equations must be solved simultaneously, of course.

For an exact solution, this system of equations will not be finite; the T operator will have an infinite number of terms, and there will be a correspondingly infinite number of auxiliary equations to fix them. We therefore truncate T to a few terms and truncate the projection $1 - P_v$ to include only enough higher Fock states to have enough equations to solve for the terms in T . The construction of these equations is aided by use of the Baker–Hausdorff expansion

$$\overline{\mathcal{P}}^- = \mathcal{P}^- + [\mathcal{P}^-, T] + \frac{1}{2}[[\mathcal{P}^-, T], T] + \dots, \quad (2.9)$$

which can be truncated to the finite number of terms relevant for the truncated projection. The restriction that T always increase particle number is critical for this truncation of the Baker–Hausdorff expansion. The exponential of T is not truncated, so that $e^T|\phi\rangle$ still contains all the higher Fock states.

The infinity of Fock states included in $e^T|\phi(\underline{P})\rangle$ makes the normalization step nontrivial. Direct computation of Z requires an infinite sum. However, we can still compute expectation values of observables, using a trick borrowed from the original coupled-cluster method [16]. Let \hat{O} be the Hermitian operator representing the observable of interest. We wish then to compute

$$\langle\hat{O}\rangle = Z\langle\phi(\underline{P})|e^{T^\dagger}\hat{O}e^T|\phi(\underline{P})\rangle. \quad (2.10)$$

To do this, we define

$$\overline{O} = e^{-T}\hat{O}e^T \quad (2.11)$$

and

$$|\tilde{\psi}(\underline{P})\rangle = Ze^{T^\dagger}e^T|\phi(\underline{P})\rangle = \sqrt{Z}e^{T^\dagger}|\psi(\underline{P})\rangle, \quad (2.12)$$

so that

$$\langle\hat{O}\rangle = \langle\tilde{\psi}(\underline{P})|\overline{O}|\phi(\underline{P})\rangle. \quad (2.13)$$

By construction, we have

$$\langle\phi(\underline{P}')|\tilde{\psi}(\underline{P})\rangle = \langle\psi(\underline{P}')|\psi(\underline{P})\rangle = \delta(\underline{P}' - \underline{P}) \quad (2.14)$$

and

$$\overline{\mathcal{P}}^{-\dagger}|\tilde{\psi}(\underline{P})\rangle = e^{T^\dagger}\mathcal{P}^-e^{-T^\dagger}\sqrt{Z}e^{T^\dagger}|\psi(\underline{P})\rangle = \sqrt{Z}e^{T^\dagger}\mathcal{P}^-|\psi(\underline{P})\rangle = \frac{M^2 + P_\perp^2}{P^+}|\tilde{\psi}(\underline{P})\rangle. \quad (2.15)$$

Thus, we can find $|\tilde{\psi}(\underline{P})\rangle$ by solving the eigenvalue problem for $\overline{\mathcal{P}}^{-\dagger} \neq \overline{\mathcal{P}}^-$, at the same mass M , and normalizing $|\psi(\underline{P})\rangle$ to the valence state $|\phi(\underline{P})\rangle$ according to Eq. (2.14). When T is truncated, $|\tilde{\psi}(\underline{P})\rangle$ must also be truncated [14], to the valence sector plus those Fock states associated with $T|\phi(\underline{P})\rangle$. This leaves a finite set of coupled equations for the functions in $|\tilde{\psi}(\underline{P})\rangle$. Also, the Baker–Hausdorff expansion of the effective operator \bar{O} can be truncated at a finite number of terms.

One useful generalization of this technique is the calculation of off-diagonal matrix elements

$$\langle\psi_1(\underline{P}_2)|\hat{O}|\psi_1(\underline{P}_1)\rangle = \sqrt{Z_1 Z_2} \langle\phi_2(\underline{P}_2)|e^{T_2^\dagger} \hat{O} e^{T_1} |\phi_1(\underline{P}_1)\rangle.$$

This can be done by defining $\bar{O}_i = e^{-T_i} \hat{O} e^{T_i}$ and $|\tilde{\psi}_i(\underline{P})\rangle = Z_i e^{T_i^\dagger} e^{T_i} |\phi_i(\underline{P})\rangle$ and then considering the two rearrangements for an Hermitian \hat{O}

$$\langle\psi_2(\underline{P}_2)|\hat{O}|\psi_1(\underline{P}_1)\rangle = \sqrt{\frac{Z_1}{Z_2}} \langle\tilde{\psi}_2(\underline{P}_2)|\bar{O}_2 e^{-T_2} e^{T_1} |\phi_1(\underline{P}_1)\rangle$$

and

$$\langle\psi_1(\underline{P}_1)|\hat{O}|\psi_2(\underline{P}_2)\rangle = \sqrt{\frac{Z_2}{Z_1}} \langle\tilde{\psi}_1(\underline{P}_1)|\bar{O}_1 e^{-T_1} e^{T_2} |\phi_2(\underline{P}_2)\rangle.$$

The unknown normalization factors cancel in the product of the first by the conjugate of the second, which yields

$$\begin{aligned} \langle\psi_2(\underline{P}_2)|\hat{O}|\psi_1(\underline{P}_1)\rangle &= \sqrt{\langle\tilde{\psi}_2(\underline{P}_2)|\bar{O}_2 e^{-T_2} e^{T_1} |\phi_1(\underline{P}_1)\rangle} \\ &\times \sqrt{\langle\tilde{\psi}_1(\underline{P}_1)|\bar{O}_1 e^{-T_1} e^{T_2} |\phi_2(\underline{P}_2)\rangle^*}. \end{aligned} \quad (2.16)$$

The correct phase can be obtained by checking one or the other of the individual matrix elements. The factors of $e^{-T_1} e^{T_2}$ and $e^{-T_2} e^{T_1}$ can be evaluated with use of power series expansions for the exponentials or the Zassenhaus expansion [20]

$$e^{T_2 - T_1} = e^{T_2} e^{-T_1} e^{\frac{1}{2}[T_2, T_1]} e^{\frac{1}{3}[T_1, [T_2, T_1]] + \frac{1}{6}[[T_2, T_1], T_2]} \dots \quad (2.17)$$

Only a finite number of terms will contribute, because the T_i only increase particle number and the initial and final states include only a finite set of Fock sectors. In the special case of an expectation value, the formula in (2.16) reduces to (2.13).

Another generalization of the technique for expectation values is to include a projection P_s of the eigenstate onto a subspace. We have in mind applications to gauge theories where the projection is onto a physical subspace of states that satisfy the gauge condition; however, in this Section we will not need to be specific. Let $|\psi_s(\underline{P})\rangle = \sqrt{Z_s} P_s e^T |\phi(\underline{P})\rangle$ be the projected state, normalized such that

$$\langle\psi_s(\underline{P}')|\psi_s(\underline{P})\rangle = \delta(\underline{P}' - \underline{P}). \quad (2.18)$$

The unprojected state is related by

$$|\psi_s(\underline{P})\rangle = \sqrt{\frac{Z_s}{Z}} P_s |\psi(\underline{P})\rangle. \quad (2.19)$$

The expectation value of an operator \hat{O} is given by

$$\langle \hat{O} \rangle_s \equiv \langle \psi_s(\underline{P}) | \hat{O} | \psi_s(\underline{P}) \rangle = Z_s \langle \phi(\underline{P}) | e^{T^\dagger} P_s^\dagger \hat{O} P_s e^T | \phi(\underline{P}) \rangle. \quad (2.20)$$

On introduction of the unprojected left-hand eigenstate $|\tilde{\psi}(\underline{P})\rangle$, this becomes

$$\langle \hat{O} \rangle_s = \frac{Z_s}{Z} \langle \tilde{\psi}(\underline{P}) | e^{-T} P_s^\dagger \hat{O} P_s e^T | \phi(\underline{P}) \rangle. \quad (2.21)$$

To obtain the ratio Z_s/Z , we assume Z_s and Z to be independent of the total momentum and use

$$\delta(\underline{P}' - \underline{P}) = \langle \psi_s(\underline{P}') | \psi_s(\underline{P}) \rangle = Z_s \langle \phi(\underline{P}') | e^{T^\dagger} P_s^\dagger P_s e^T | \phi(\underline{P}) \rangle = \frac{Z_s}{Z} \langle \tilde{\psi}(\underline{P}') | e^{-T} P_s^\dagger P_s e^T | \phi(\underline{P}) \rangle. \quad (2.22)$$

Integration over \underline{P}' then yields

$$\frac{Z}{Z_s} = \int d\underline{P}' \langle \tilde{\psi}(\underline{P}') | e^{-T} P_s^\dagger P_s e^T | \phi(\underline{P}) \rangle. \quad (2.23)$$

We therefore obtain

$$\langle \hat{O} \rangle_s = \frac{\langle \tilde{\psi}(\underline{P}) | e^{-T} P_s^\dagger \hat{O} P_s e^T | \phi(\underline{P}) \rangle}{\int d\underline{P}' \langle \tilde{\psi}(\underline{P}') | e^{-T} P_s^\dagger P_s e^T | \phi(\underline{P}) \rangle}. \quad (2.24)$$

The remaining matrix elements are computed from the known $|\phi(\underline{P})\rangle$ and $|\tilde{\psi}(\underline{P})\rangle$ by first expanding the exponentials, keeping the total number of T factors consistent with the truncation of T , carrying out any contractions of annihilation and creation operators, and then applying the projections P_s and P_s^\dagger . A Baker–Hausdorff expansion is not useful here, because of the projections, but the truncated expansion of the exponentials is equivalent to a truncated Baker–Hausdorff expansion.

III. AN APPLICATION TO QED

To show how the LFCC method works for a gauge theory, we consider the dressed-electron state in QED, truncated to exclude positrons. The theory is regulated by one Pauli–Villars (PV) electron, with coupling coefficient $\beta_1 = 1$, and two PV photons, with coupling coefficients ξ_1 and ξ_2 . Additional PV fields are not needed if positrons are absent [21]. The construction of the covariant-gauge light-front Hamiltonian is discussed in Appendix A, with additional details given in [9]. The valence state is

$$|\phi_a^\pm(\underline{P})\rangle = \sum_i z_{ai}^\pm b_{i\pm}^\dagger(\underline{P}) |0\rangle, \quad (3.1)$$

and the LFCC eigenstate is

$$|\psi_a^\sigma(\underline{P})\rangle = \sqrt{Z} e^T |\phi_a^\sigma(\underline{P})\rangle. \quad (3.2)$$

We truncate the T operator to just photon emission from an electron

$$T = \sum_{ijl\sigma s\lambda} \int dy d\vec{k}_\perp \int \frac{dp}{\sqrt{16\pi^3}} \sqrt{p^+} t_{ijl}^{\sigma s\lambda}(y, \vec{k}_\perp) a_{l\lambda}^\dagger(y, \vec{k}_\perp; \underline{p}) b_{js}^\dagger(1-y, -\vec{k}_\perp; \underline{p}) b_{i\sigma}(\underline{p}). \quad (3.3)$$

Surprisingly, this will introduce as much physics as a two-photon truncation of the Fock space [19]. The resulting effective Hamiltonian $\overline{\mathcal{P}}^-$ is derived in Appendix B.

A. Right-hand eigenvalue problem

The projection (2.7) of the eigenvalue problem onto the valence sector yields, for the effective Hamiltonian in (B17),

$$m_i^2 z_{ai}^\pm + \sum_j I_{ij} z_{aj}^\pm = M_a^2 z_{ai}^\pm, \quad (3.4)$$

with $a = 0, 1$ and M_a the a th eigenmass. The self-energy I_{ij} is defined in (B10). Clearly, the amplitudes z_{ai}^\pm are actually independent of spin, and the spin index \pm can be dropped. We also have left eigenbras of $P_v \overline{\mathcal{P}}^- P_v$,

$$\langle \tilde{\phi}_a^\pm(\underline{P}) | = \langle 0 | \sum_i \tilde{z}_{ai} b_{i\pm}(\underline{P}), \quad (3.5)$$

for which the amplitudes satisfy

$$m_i^2 \tilde{z}_{ai} + \sum_j (-1)^{i+j} I_{ji} \tilde{z}_{aj} = M_a^2 \tilde{z}_{ai}. \quad (3.6)$$

These amplitudes are also independent of spin.

The left and right valence eigenvectors for different eigenvalues are orthogonal. The normalizations are chosen to satisfy

$$\langle \tilde{\phi}_a^{\sigma'}(\underline{P}') | \phi_b^\sigma(\underline{P}) \rangle = (-1)^a \delta_{ab} \delta_{\sigma\sigma'} \delta(\underline{P}' - \underline{P}) \quad (3.7)$$

Notice that the $a = 1$ state has negative norm. We then have

$$\sum_i (-1)^i \tilde{z}_{ai} z_{bi} = (-1)^a \delta_{ab}. \quad (3.8)$$

We also have an identity matrix in the valence sector

$$\sum_a (-1)^a z_{ai} \tilde{z}_{aj} = (-1)^i \delta_{ij}. \quad (3.9)$$

Projection of the eigenvalue problem onto the one-electron/one-photon sector gives

$$\begin{aligned} & \sum_i (-1)^i z_{ai} \left\{ h_{ijl}^{\pm s\lambda}(y, \vec{k}_\perp) + \frac{1}{2} V_{ijl}^{\pm s\lambda}(y, \vec{k}_\perp) \right. \\ & + \left[\frac{m_j^2 + k_\perp^2}{1-y} + \frac{\mu_{l\lambda}^2 + k_\perp^2}{y} - m_i^2 \right] t_{ijl}^{\pm s\lambda}(y, \vec{k}_\perp) \\ & \left. + \frac{1}{2} \sum_{i'} \frac{I_{ji'}}{1-y} t_{ii'l}^{\pm s\lambda}(y, \vec{k}_\perp) - \sum_{j'} (-1)^{i+j'} t_{j'jl}^{\pm s\lambda}(y, \vec{k}_\perp) I_{j'i} \right\} = 0. \end{aligned} \quad (3.10)$$

To partially diagonalize in flavor, we define

$$C_{abl}^{\sigma s\lambda}(y, \vec{k}_\perp) \equiv \sum_{ij} (-1)^{i+j} z_{ai} \tilde{z}_{bj} t_{ijl}^{\sigma s\lambda}(y, \vec{k}_\perp). \quad (3.11)$$

With analogous definitions for the vertex functions H and the vertex-loop correction V , and with

$$I_{bb'} \equiv (-1)^{b'} \sum_{ij} (-1)^i \tilde{z}_{bi} \tilde{z}_{b'j} I_{ij}, \quad (3.12)$$

we have

$$\left[M_a^2 - \frac{M_b^2 + k_\perp^2}{1-y} - \frac{\mu_{l\lambda}^2 + k_\perp^2}{y} \right] C_{abl}^{\sigma s \lambda}(y, \vec{k}_\perp) = H_{abl}^{\sigma s \lambda}(y, \vec{k}_\perp) + \frac{1}{2} \left[V_{abl}^{\sigma s \lambda}(y, \vec{k}_\perp) - \sum_{b'} \frac{I_{bb'}}{1-y} C_{ab'l}^{\sigma s \lambda}(y, \vec{k}_\perp) \right]. \quad (3.13)$$

Here the eigenmass M_b has replaced the bare mass m_j in a natural way, without invocation of a sector-dependent parameterization. These equations are to be solved simultaneously with the valence-sector equations, (3.4) and (3.6). To facilitate the calculation, the self-energy contribution $I_{bb'}$ and the vertex correction $V_{abl}^{\sigma s \lambda}$ can be expressed directly in terms of the wave functions $C_{abl}^{\sigma s \lambda}$ as

$$I_{bb'} = (-1)^{b'} \sum_{als\lambda} (-1)^{a+l} \epsilon^\lambda \int \frac{dy d\vec{k}_\perp}{16\pi^3} \tilde{H}_{bal}^{\pm s \lambda*}(y, \vec{k}_\perp) C_{b'al}^{\pm s \lambda}(y, \vec{k}_\perp) \quad (3.14)$$

and

$$V_{abl}^{\sigma s \lambda}(y, \vec{k}_\perp) = \sum_{a'b'l'\sigma's'\lambda'} (-1)^{a'+b'+l'} \epsilon^{\lambda'} \int \frac{dy' d\vec{k}'_\perp}{16\pi^3} \frac{\theta(1-y-y')}{\sqrt{(1-y')(1-y)^3}} \times \tilde{H}_{bb'l'}^{\sigma s' \lambda'*}(\frac{y'}{1-y}, \vec{k}'_\perp + \frac{y'}{1-y} \vec{k}_\perp) C_{a'b'l}^{\sigma' s' \lambda}(\frac{y}{1-y}, \vec{k}_\perp + \frac{y}{1-y} \vec{k}'_\perp) C_{aa'l'}^{\sigma \sigma' \lambda'}(y', \vec{k}'_\perp), \quad (3.15)$$

where

$$\tilde{H}_{abl}^{\sigma s \lambda}(y, \vec{k}_\perp) = \sum_{ij} (-1)^{i+j} \tilde{z}_{ai} \tilde{z}_{bj} h_{ijl}^{\sigma s \lambda}(y, \vec{k}_\perp). \quad (3.16)$$

The original self-energy contribution I_{ij} , defined in (B10), can be obtained as

$$I_{ij} = (-1)^j \sum_{bb'} (-1)^b \tilde{z}_{bi} \tilde{z}_{b'j} I_{bb'}. \quad (3.17)$$

The wave functions $C_{abl}^{\sigma s \lambda}$ for different J_z index σ are then seen to be related in the same pattern as the vertex functions $H_{abl}^{\sigma s \lambda}$. Since the spin and polarization dependence is not affected by the flavor diagonalizations, this pattern can be read from the structure of the fundamental vertex functions $h_{abl}^{\sigma s \lambda}$ given in (A22). We find

$$C_{abl}^{-+\lambda} = C_{abl}^{+-\lambda*}, \quad C_{abl}^{--\lambda} = -C_{abl}^{++\lambda*}, \quad \text{for } \lambda = \pm, \quad (3.18)$$

$$C_{abl}^{-+\lambda} = -C_{abl}^{+-\lambda*}, \quad C_{abl}^{--\lambda} = C_{abl}^{++\lambda*}, \quad \text{for } \lambda = 0, 3. \quad (3.19)$$

B. Left-hand eigenvalue problem

To obtain the left-hand eigenstates, for use in computation of matrix elements, we also need to solve

$$\overline{\mathcal{P}}^{-\dagger} |\tilde{\psi}_a^\sigma(\underline{P})\rangle = \frac{M_a^2 + P_\perp^2}{P^+} |\tilde{\psi}_a^\sigma(\underline{P})\rangle, \quad (3.20)$$

with M_a fixed and $\overline{\mathcal{P}}^-$ simplified by using $t_{ijl}^{\sigma s \lambda}$ as a solution to Eq. (3.10), for which the curly bracket in the third term of (B17) is zero. The conjugate is then

$$\begin{aligned} \overline{\mathcal{P}}^{-\dagger} = & \sum_{ijs} (-1)^j \int d\underline{p} \left[\delta_{ij} \frac{m_i^2 + p_\perp^2}{p^+} + \frac{I_{ij}}{p^+} \right] b_{js}^\dagger(\underline{p}) b_{is}(\underline{p}) \\ & + \sum_{l\lambda} (-1)^l \epsilon^\lambda \int d\underline{p} \frac{\mu_{l\lambda}^2 + p_\perp^2}{p^+} a_{l\lambda}^\dagger(\underline{p}) a_{l\lambda}(\underline{p}) \\ & + \sum_{ijl\sigma s \lambda} \int dy d\vec{k}_\perp \int \frac{d\underline{p}}{\sqrt{16\pi^3 p^+}} h_{ijl}^{\sigma s \lambda}(y, \vec{k}_\perp) a_{l\lambda}^\dagger(y, \vec{k}_\perp; \underline{p}) b_{js}^\dagger(1-y, -\vec{k}_\perp; \underline{p}) b_{i\sigma}(\underline{p}) \\ & + \sum_{ijl\sigma s \lambda} (-1)^j \sum_{i'l'\sigma'\lambda'} \int dy d\vec{k}_\perp \int \frac{d\underline{p}}{\sqrt{16\pi^3 p^+}} \int dy' d\vec{k}'_\perp \int \frac{d\underline{p}'}{\sqrt{16\pi^3}} \sqrt{p'^+} \\ & \quad \times \delta((1-y)p^+ - (1-y')p'^+) \delta((1-y)\vec{p}_\perp - \vec{k}_\perp - (1-y')\vec{p}'_\perp + \vec{k}'_\perp) \\ & \quad \times h_{ijl}^{\sigma s \lambda}(y, \vec{k}_\perp) t_{i'jl'}^{\sigma' s \lambda' *} (y', \vec{k}'_\perp) a_{l\lambda}^\dagger(y, \vec{k}_\perp; \underline{p}) b_{i'\sigma'}^\dagger(\underline{p}') b_{i\sigma}(\underline{p}) a_{l'\lambda'}(y', \vec{k}'_\perp; \underline{p}') \\ & - \sum_{ijl\sigma s \lambda} (-1)^i \sum_{j'l's'\lambda'} \int dy d\vec{k}_\perp \int \frac{d\underline{p}}{16\pi^3} \int dy' d\vec{k}'_\perp h_{ijl}^{\sigma s \lambda}(y, \vec{k}_\perp) t_{ij'l'}^{\sigma s \lambda' *} (y', \vec{k}'_\perp) \\ & \quad \times a_{l\lambda}^\dagger(y, \vec{k}_\perp; \underline{p}) b_{js}^\dagger(1-y, -\vec{k}_\perp; \underline{p}) b_{j's'}(1-y', -\vec{k}'_\perp; \underline{p}) a_{l'\lambda'}(y', \vec{k}'_\perp; \underline{p}'). \end{aligned} \quad (3.21)$$

The truncated left-hand eigenvector is

$$\begin{aligned} |\tilde{\psi}_a^\sigma(\underline{P})\rangle &= |\tilde{\phi}_a^\sigma(\underline{P})\rangle \\ &+ \sum_{jls\lambda} \int dy d\vec{k}_\perp \sqrt{\frac{P^+}{16\pi^3}} l_{ajl}^{\sigma s \lambda}(y, \vec{k}_\perp) a_{l\lambda}^\dagger(y, \vec{k}_\perp; \underline{P}) b_{js}^\dagger(1-y, -\vec{k}_\perp; \underline{P}) |0\rangle, \end{aligned} \quad (3.22)$$

where $|\tilde{\phi}_a^\pm(\underline{P})\rangle = \sum_i \tilde{z}_{ai} b_{i\pm}^\dagger(\underline{P}) |0\rangle$ is fixed.

We diagonalize in flavor, defining

$$D_{abl}^{\sigma s \lambda}(y, \vec{k}_\perp) \equiv \sum_j (-1)^j z_{bj}^s l_{ajl}^{\sigma s \lambda}(y, \vec{k}_\perp), \quad (3.23)$$

$$J_{ba}^\sigma = (-1)^b \sum_{b'l's\lambda} (-1)^{b'+l} \epsilon^\lambda \int \frac{dy d\vec{k}_\perp}{16\pi^3} C_{bb'l}^{\sigma s \lambda *} (y, \vec{k}_\perp) D_{ab'l}^{\sigma s \lambda}(y, \vec{k}_\perp), \quad (3.24)$$

and

$$\begin{aligned} W_{abl}^{\sigma s \lambda}(y, \vec{k}_\perp) &= \sum_{a'b'l's'\sigma'\lambda'} (-1)^{a'+b'+l'} \epsilon^{\lambda'} \int \frac{dy' d\vec{k}'_\perp}{16\pi^3} \frac{\theta(1-y-y')}{\sqrt{(1-y')^3(1-y)}} \\ &\quad \times C_{bb'l'}^{\sigma s \lambda' *} \left(\frac{y'}{1-y}, \vec{k}'_\perp + \frac{y'}{1-y} \vec{k}_\perp \right) \tilde{H}_{a'b'l}^{\sigma' s' \lambda} \left(\frac{y}{1-y'}, \vec{k}_\perp + \frac{y}{1-y'} \vec{k}'_\perp \right) D_{aa'l'}^{\sigma \sigma' \lambda'}(y', \vec{k}'_\perp). \end{aligned} \quad (3.25)$$

This yields

$$\left[M_a^2 - \frac{M_b^2 + k_\perp^2}{1-y} - \frac{\mu_{l\lambda}^2 + k_\perp^2}{y} \right] D_{abl}^{\sigma s \lambda}(y, \vec{k}_\perp) = \tilde{H}_{abl}^{\sigma s \lambda}(y, \vec{k}_\perp) + W_{abl}^{\sigma s \lambda}(y, \vec{k}_\perp) - \sum_{b'} J_{b'a}^\sigma \tilde{H}_{b'bl}^{\sigma s \lambda}(y, \vec{k}_\perp). \quad (3.26)$$

The right-hand wave functions $C_{abl}^{\sigma s \lambda}$ are input to this set of equations. The left-hand wave functions $D_{abl}^{\sigma s \lambda}$ for different J_z index are related in the same way as the right-hand wave functions:

$$D_{abl}^{-+\lambda} = D_{abl}^{+-\lambda*}, \quad D_{abl}^{--\lambda} = -D_{abl}^{++\lambda*}, \quad \text{for } \lambda = \pm, \quad (3.27)$$

$$D_{abl}^{-+\lambda} = -D_{abl}^{+-\lambda*}, \quad D_{abl}^{--\lambda} = D_{abl}^{++\lambda*}, \quad \text{for } \lambda = 0, 3. \quad (3.28)$$

Both sets of equations, left and right, require numerical techniques for their solution.

IV. ANOMALOUS MAGNETIC MOMENT

We extract the anomalous magnetic moment of the electron from the spin-flip matrix element of the current

$$J^+ = \bar{\psi} \gamma^+ \psi = 2\psi_+ \psi_+ = 2 \sum_{ij} \psi_{i+} \psi_{j+}. \quad (4.1)$$

The ψ_{i+} are given by (A4). For our normalization, the general matrix element is [22]

$$16\pi^3 \langle \psi_a^\sigma(\underline{P} + \underline{q}) | J^+(0) | \psi_a^\pm(\underline{P}) \rangle = 2\delta_{\sigma\pm} F_1(q^2) \pm \frac{q^1 \pm iq^2}{M_a} \delta_{\sigma\mp} F_2(q^2). \quad (4.2)$$

We compute in the Drell-Yan frame [23], where $q^+ = 0$ and $P'^+ = P^+$, and there are no pair contributions, so that

$$J^+(0) = 2 \sum_{ijs} \int \frac{d\underline{p}'}{\sqrt{16\pi^3}} \int \frac{d\underline{p}}{\sqrt{16\pi^3}} b_{is}^\dagger(\underline{p}') b_{js}(\underline{p}). \quad (4.3)$$

We can apply the formalism for the projected expectation value (2.24), with the projection P_s being the projection onto the physical subspace with the two transverse polarizations $\lambda = 1, 2$. The current matrix element is given by

$$\langle \psi_a^\sigma(\underline{P} + \underline{q}) | J^+(0) | \psi_a^\pm(\underline{P}) \rangle = \frac{\langle \tilde{\psi}_a^\sigma(\underline{P} + \underline{q}) | e^{-T} P_s^\dagger J^+(0) P_s e^T | \phi_a^\pm(\underline{P}) \rangle}{\int d\underline{P}' \langle \tilde{\psi}_a^\pm(\underline{P}') | e^{-T} P_s^\dagger P_s e^T | \phi_a^\pm(\underline{P}) \rangle}. \quad (4.4)$$

Note that T is common to both spin projections $\sigma = \pm$ and flavor eigenstates $a = 0, 1$; therefore, a general expression for off-diagonal matrix elements is not needed here.

For the chosen truncation, $e^{\pm T}$ can be replaced by $1 \pm T$ and only terms up to first order in T are to be kept. On substitution of (3.22) for $|\tilde{\psi}_a^\sigma\rangle$, we have

$$\begin{aligned} \langle \tilde{\psi}_a^\sigma(\underline{P} + \underline{q}) | e^{-T} P_s^\dagger J^+(0) P_s e^T | \phi_a^\pm(\underline{P}) \rangle &= \frac{2}{16\pi^3} \left[\delta_{\sigma\pm} (\tilde{z}_{a0} - \tilde{z}_{a1}) (z_{a0} - z_{a1}) \right. \\ &+ \sum_{ijj'ls} (-1)^{i+j+j'+l} \int \frac{dy d\vec{k}_\perp}{16\pi^3} \left\{ \sum_{\lambda=\pm} l_{ajl}^{\sigma s \lambda*}(y, \vec{k}_\perp - y\vec{q}_\perp) t_{ij'l}^{\pm s \lambda}(y, \vec{k}_\perp) z_{ai} \right. \\ &\quad \left. \left. - \sum_{\lambda=0}^3 \epsilon^\lambda l_{ajl}^{\sigma s \lambda*}(y, \vec{k}_\perp) t_{ij'l}^{\pm s \lambda}(y, \vec{k}_\perp) z_{aj'} \right\} \right] \end{aligned} \quad (4.5)$$

for the numerator and

$$\langle \tilde{\psi}_a^\pm(\underline{P}') | e^{-T} P_s^\dagger P_s e^T | \phi_a^\pm(\underline{P}) \rangle = \delta(\underline{P}' - \underline{P}) \left[1 - \sum_{ijls} (-1)^{i+j+l} \sum_{\lambda=0,3} \epsilon^\lambda l_{ajl}^{\pm s \lambda*}(y, \vec{k}_\perp) t_{ij'l}^{\pm s \lambda}(y, \vec{k}_\perp) z_{ai} \right] \quad (4.6)$$

for the denominator.

We can then extract the form factors (for $a = 0$) as

$$\begin{aligned} F_1(q^2) &= \frac{1}{\mathcal{N}} \left[(\tilde{z}_{00} - \tilde{z}_{01}) (z_{00} - z_{01}) \right. \\ &+ \sum_{ijj'ls} (-1)^{i+j+j'+l} \int \frac{dy d\vec{k}_\perp}{16\pi^3} \left\{ \sum_{\lambda=\pm} l_{0il}^{\pm s \lambda*}(y, \vec{k}_\perp - y\vec{q}_\perp) t_{j'jl}^{\pm s \lambda}(y, \vec{k}_\perp) z_{0j'} \right. \\ &\quad \left. \left. - \sum_{\lambda=0}^3 \epsilon^\lambda l_{0j'l}^{\pm s \lambda*}(y, \vec{k}_\perp) t_{ij'l}^{\pm s \lambda}(y, \vec{k}_\perp) z_{0j} \right\} \right] \end{aligned} \quad (4.7)$$

and

$$\begin{aligned} F_2(q^2) &= \pm \frac{2M_0}{q^1 \pm iq^2} \sum_{ijl} (-1)^{i+j+l} \sum_{j's} (-1)^{j'} \int \frac{dy d\vec{k}_\perp}{16\pi^3} \\ &\quad \times \frac{1}{\mathcal{N}} \left\{ \sum_{\lambda=\pm} l_{0il}^{\mp s \lambda*}(y, \vec{k}_\perp - y\vec{q}_\perp) t_{j'jl}^{\pm s \lambda}(y, \vec{k}_\perp) z_{0j'} \right. \\ &\quad \left. - \sum_{\lambda=0}^3 \epsilon^\lambda l_{0j'l}^{\mp s \lambda*}(y, \vec{k}_\perp) t_{ij'l}^{\pm s \lambda}(y, \vec{k}_\perp) z_{0j} \right\}, \end{aligned} \quad (4.8)$$

with

$$\mathcal{N} = 1 - \sum_{ijls} (-1)^{i+j+l} \sum_{\lambda=0,3} \epsilon^\lambda \int \frac{dy d\vec{k}_\perp}{16\pi^3} l_{0jl}^{\pm s \lambda*}(y, \vec{k}_\perp) t_{ij'l}^{\pm s \lambda}(y, \vec{k}_\perp) z_{0i}. \quad (4.9)$$

However, for opposite spins, l and t are orthogonal, because the azimuthal integration yields

$$\int dy d\vec{k}_\perp l_{aj'l}^{\mp s \lambda*}(y, \vec{k}_\perp) t_{ij'l}^{\pm s \lambda}(y, \vec{k}_\perp) = 0. \quad (4.10)$$

This eliminates the second term in F_2 . In the limit that $q^2 \rightarrow 0$, the first term can be rewritten as a derivative, as shown in [22], to obtain $a_e = F_2(0)$ as

$$a_e = \pm \frac{M_0}{\mathcal{N}} \sum_{ijj'ls} (-1)^{i+j+j'+l} \sum_{\lambda=\pm} \int \frac{dy d\vec{k}_\perp}{16\pi^3} y l_{0il}^{\mp s\lambda*}(y, \vec{k}_\perp) \left(\frac{\partial}{\partial k^1} \mp i \frac{\partial}{\partial k^2} \right) t_{j'jl}^{\pm s\lambda}(y, \vec{k}_\perp) z_{0j'}. \quad (4.11)$$

In terms of the wave functions, $C_{abl}^{\sigma s\lambda}$ and $D_{abl}^{\sigma s\lambda}$, we have

$$a_e = \pm \frac{M_0}{\mathcal{N}} \sum_{ijabls} (-1)^{i+j+a+b+l} \tilde{z}_{ai} z_{bj} \sum_{\lambda=\pm} \int \frac{dy d\vec{k}_\perp}{16\pi^3} y D_{0al}^{\mp s\lambda*}(y, \vec{k}_\perp) \left(\frac{\partial}{\partial k^1} \mp i \frac{\partial}{\partial k^2} \right) C_{obl}^{\pm s\lambda}(y, \vec{k}_\perp). \quad (4.12)$$

and

$$\mathcal{N} = 1 - \sum_{bls} (-1)^{b+l} \sum_{\lambda=0,3} \epsilon^\lambda \int \frac{dy d\vec{k}_\perp}{16\pi^3} D_{obl}^{\pm s\lambda*}(y, \vec{k}_\perp) C_{obl}^{\pm s\lambda}(y, \vec{k}_\perp). \quad (4.13)$$

In the limit of infinite PV masses, and with $M_0 = m_e$ the electron mass,

$$F_1(q^2) = \frac{1}{\mathcal{N}} \left[1 + \sum_s \int \frac{dy d\vec{k}_\perp}{16\pi^3} \left\{ \sum_{\lambda=\pm} l_{000}^{\pm s\lambda*}(y, \vec{k}_\perp - y\vec{q}_\perp) t_{000}^{\pm s\lambda}(y, \vec{k}_\perp) - \sum_{\lambda=0}^3 \epsilon^\lambda l_{000}^{\pm s\lambda*}(y, \vec{k}_\perp) t_{000}^{\pm s\lambda}(y, \vec{k}_\perp) \right\} \right] \quad (4.14)$$

and

$$F_2(q^2) = \pm \frac{2m_e}{q^1 \pm iq^2} \frac{1}{\mathcal{N}} \sum_s \sum_{\lambda=\pm} \int \frac{dy d\vec{k}_\perp}{16\pi^3} l_{000}^{\mp s\lambda*}(y, \vec{k}_\perp - y\vec{q}_\perp) t_{000}^{\pm s\lambda}(y, \vec{k}_\perp), \quad (4.15)$$

with

$$\mathcal{N} = 1 - \sum_s \sum_{\lambda=0,3} \epsilon^\lambda \int \frac{dy d\vec{k}_\perp}{16\pi^3} l_{000}^{\pm s\lambda*}(y, \vec{k}_\perp) t_{000}^{\pm s\lambda}(y, \vec{k}_\perp). \quad (4.16)$$

In the $q^2 \rightarrow 0$ limit, we have $F_1(0) = 1$ and

$$a_e = \pm \frac{m_e}{\mathcal{N}} \sum_s \sum_{\lambda=\pm} \int \frac{dy d\vec{k}_\perp}{16\pi^3} y l_{000}^{\mp s\lambda*}(y, \vec{k}_\perp) \left(\frac{\partial}{\partial k^1} \mp i \frac{\partial}{\partial k^2} \right) t_{000}^{\pm s\lambda}(y, \vec{k}_\perp). \quad (4.17)$$

The leading perturbative result is

$$t_{000}^{\sigma s\lambda}(y, \vec{k}_\perp) = l_{000}^{\sigma s\lambda}(y, \vec{k}_\perp) = \frac{h_{000}^{\sigma s\lambda}(y, \vec{k}_\perp)}{m_e^2 - \frac{m_e^2 + k_\perp^2}{1-y} - \frac{\mu_{0\lambda}^2 + k_\perp^2}{y}}. \quad (4.18)$$

When μ_0 goes to zero, we find $\mathcal{N} = 1 + \mathcal{O}(\alpha^2)$ and

$$a_e = \frac{\alpha}{2\pi} + \mathcal{O}(\alpha^2), \quad (4.19)$$

which agrees with the Schwinger result [24]. The result for the normalization factor \mathcal{N} depends on the fact that $h_{ijl}^{\sigma s\lambda}$ is the same for $\lambda = 0$ and 3 when the photon mass μ_l is zero, as found in (A22), so that the leading $\mathcal{O}(\alpha)$ contributions to the sum over λ cancel.

V. SUMMARY

We have applied the LFCC method [14] to QED in an arbitrary covariant gauge. The anomalous moment of the electron is given by Eq. (4.12), which must be evaluated with use of the left-hand and right-hand functions $D_{ijl}^{\sigma s \lambda}$ and $C_{ijl}^{\sigma s \lambda}$. These functions are obtained as solutions of the corresponding eigenvalue problems (3.26) and (3.13). The leading perturbative contribution to the anomalous moment is verified to be the standard Schwinger term; a nonperturbative solution requires numerical methods. To carry out this construction, we have extended our analysis of arbitrary gauges [9] to include a specific choice of projection onto physical states, as discussed at the end of Appendix A, and have extended the calculation of LFCC matrix elements [14] to include such projections, as expressed in Eq. (2.24).

This provides an extensive test of the LFCC method in a gauge theory regulated by PV fields. However, the method is much more general than this; it should be applicable to any regulated light-front Hamiltonian. The avoidance of a Fock-space truncation then provides several benefits, with the absence of Fock-sector dependence and spectator dependence and of the uncanceled divergences that can result from them. The method allows for systematic improvement of a calculation, through the addition of terms to the exponentiated operator T . Thus a natural next step is to include one or more positron terms, to study the dressed-photon state [21], pair contributions to the dressed-electron state, and positronium.

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Appendix A: Light-front QED in an arbitrary gauge

Here we summarize a formulation of light-front QED regulated by the inclusion of Pauli-Villars (PV) electrons and photons [26] and quantized in an arbitrary covariant gauge. Details can be found in [9]. We write the Lagrangian as

$$\begin{aligned} \mathcal{L} = & \sum_{i=0}^2 (-1)^i \left[-\frac{1}{4} F_i^{\mu\nu} F_{i,\mu\nu} + \frac{1}{2} \mu_i^2 A_i^\mu A_{i\mu} - \frac{1}{2} \zeta (\partial^\mu A_{i\mu})^2 \right] \\ & + \sum_{i=0}^2 (-1)^i \bar{\psi}_i (i\gamma^\mu \partial_\mu - m_i) \psi_i - e \bar{\psi} \gamma^\mu \psi A_\mu, \end{aligned} \quad (\text{A1})$$

with

$$\psi = \sum_{i=0}^2 \sqrt{\beta_i} \psi_i, \quad A_\mu = \sum_{i=0}^2 \sqrt{\xi_i} A_{i\mu}, \quad F_{i\mu\nu} = \partial_\mu A_{i\nu} - \partial_\nu A_{i\mu}. \quad (\text{A2})$$

The index i is zero for physical fields and 1 or 2 for PV fields. The coupling coefficients β_i and ξ_i are constrained to satisfy $\beta_0 = 1$, $\xi_0 = 1$, and

$$\sum_{i=0}^2 (-1)^i \xi_i = 0, \quad \sum_{i=0}^2 (-1)^i \beta_i = 0, \quad (\text{A3})$$

and to reproduce the correct chiral symmetry in the limit of a massless electron [8], and to guarantee a massless eigenstate for the photon [21].

The fermion fields ψ_i are decomposed into dynamical and nondynamical parts $\psi_{i\pm} \equiv \Lambda_{\pm}\psi_i$ by the complementary projections $\Lambda_{\pm} \equiv \gamma^0\gamma^{\pm}/2$ [3, 25]. The dynamical part is written

$$\psi_{i+} = \frac{1}{\sqrt{16\pi^3}} \sum_s \int d\underline{k} \chi_s \left[b_{is}(\underline{k}) e^{-i\underline{k}\cdot\underline{x}} + d_{i,-s}^{\dagger}(\underline{k}) e^{i\underline{k}\cdot\underline{x}} \right], \quad (\text{A4})$$

with

$$\chi_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \chi_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad (\text{A5})$$

and

$$\{b_{is}(\underline{k}), b_{i's'}^{\dagger}(\underline{k}')\} = (-1)^i \delta_{ii'} \delta_{ss'} \delta(\underline{k} - \underline{k}'), \quad (\text{A6})$$

$$\{d_{is}(\underline{k}), d_{i's'}^{\dagger}(\underline{k}')\} = (-1)^i \delta_{ii'} \delta_{ss'} \delta(\underline{k} - \underline{k}'). \quad (\text{A7})$$

The nondynamical part satisfies the constraint

$$\begin{aligned} i(-1)^i \partial_- \psi_{i-} + e A_- \sqrt{\beta_i} \sum_j \psi_{j-} \\ = (i\gamma^0 \gamma^{\perp}) \left[(-1)^i \partial_{\perp} \psi_{i+} - ie A_{\perp} \sqrt{\beta_i} \sum_j \psi_{j+} \right] - (-1)^i m_i \gamma^0 \psi_{i+}. \end{aligned} \quad (\text{A8})$$

However, the constraint for the sum that enters the interaction becomes simply

$$i\partial_- \psi_- = (i\gamma^0 \gamma^{\perp}) \partial_{\perp} \psi_+ - \gamma^0 \sum_i \sqrt{\beta_i} m_i \psi_{i+}, \quad (\text{A9})$$

where the photon field does not appear. The constraint can then be trivially solved and the nondynamical field removed from the Lagrangian.

A vector field of mass μ_l is written as

$$\begin{aligned} A_{l\mu}(x) = \int \frac{d\underline{k}}{\sqrt{16\pi^3 k^+}} \left\{ \sum_{\lambda=1}^3 e_{\mu}^{(\lambda)}(\underline{k}) \left[a_{l\lambda}(\underline{k}) e^{-i\underline{k}\cdot\underline{x}} + a_{l\lambda}^{\dagger}(\underline{k}) e^{i\underline{k}\cdot\underline{x}} \right] \right. \\ \left. + e_{\mu}^{(0)}(\underline{k}) \left[a_{l0}(\underline{k}) e^{-i\tilde{\underline{k}}\cdot\underline{x}} + a_{l0}^{\dagger}(\underline{k}) e^{i\tilde{\underline{k}}\cdot\underline{x}} \right] \right\}, \end{aligned} \quad (\text{A10})$$

with $\tilde{\underline{k}}$ a four-vector associated with a different mass $\tilde{\mu}_l \equiv \mu_l/\sqrt{\zeta}$, such that

$$\tilde{\underline{k}} = \underline{k}, \quad \tilde{k}^- = (k_{\perp}^2 + \tilde{\mu}_l^2)/k^+. \quad (\text{A11})$$

This allows the field to satisfy the Euler–Lagrange equation, while the fourth polarization $\lambda = 0$ does not satisfy the gauge condition $\partial \cdot A_l = 0$ [9]. The polarization vectors are defined by

$$e^{(1,2)}(\underline{k}) = (0, 2\hat{e}_{1,2} \cdot \vec{k}_{\perp}/k^+, \hat{e}_{1,2}), \quad (\text{A12})$$

$$e^{(3)}(\underline{k}) = ((k_{\perp}^2 - \mu_l^2)/k^+, k^+, \vec{k}_{\perp})/\mu_l, \quad (\text{A13})$$

$$e^{(0)}(\underline{k}) = \tilde{k}/\mu_l = ((k_{\perp}^2 + \tilde{\mu}_l^2)/k^+, k^+, \vec{k}_{\perp})/\mu_l, \quad (\text{A14})$$

and the commutation relations are

$$[a_{l\lambda}(\underline{k}), a_{l'\lambda'}^\dagger(\underline{k}')] = (-1)^l \delta_{ll'} \epsilon^\lambda \delta_{\lambda\lambda'} \delta(\underline{k} - \underline{k}'), \quad (\text{A15})$$

with $\epsilon = (-1, 1, 1, 1)$ the metric signature for the physical photon.

For the purpose of making J_z -conservation explicit, it is convenient to introduce circular polarizations

$$e^{(\pm)} = \mp \frac{1}{\sqrt{2}} (e^{(1)} \pm i e^{(2)}) \quad (\text{A16})$$

and the associated creation and annihilation operators

$$a_{l\pm}^\dagger = \mp \frac{1}{\sqrt{2}} (a_{l1}^\dagger \pm i a_{l2}^\dagger), \quad a_{l\pm} = \mp \frac{1}{\sqrt{2}} (a_{l1} \mp i a_{l2}). \quad (\text{A17})$$

These operators satisfy the same commutation relations (A15), with λ now taking the values \pm instead of 1 and 2, and with $\epsilon^\pm \equiv 1$. Sums over λ will, in general, include 0, \pm , and 3.

The light-front Hamiltonian is then given by

$$\mathcal{P}^- = \mathcal{P}_{0a}^- + \mathcal{P}_{0b}^- + \mathcal{P}_{\text{int}}^-, \quad (\text{A18})$$

with

$$\mathcal{P}_{0a}^- = \sum_{l\lambda} (-1)^l \epsilon^\lambda \int d\underline{p} \frac{\mu_{l\lambda}^2 + p_\perp^2}{p^+} a_{l\lambda}^\dagger(\underline{p}) a_{l\lambda}(\underline{p}), \quad (\text{A19})$$

$$\mathcal{P}_{0b}^- = \sum_{is} (-1)^i \int d\underline{p} \frac{m_i^2 + p_\perp^2}{p^+} b_{is}^\dagger(\underline{p}) b_{is}(\underline{p}), \quad (\text{A20})$$

and, if antifermion terms are neglected,

$$\begin{aligned} \mathcal{P}_{\text{int}}^- = & \sum_{ijl\sigma s\lambda} \int dy d\vec{k}_\perp \int \frac{d\underline{p}}{\sqrt{16\pi^3 p^+}} \\ & \times \left\{ h_{ijl}^{\sigma s\lambda}(y, \vec{k}_\perp) a_{l\lambda}^\dagger(y, \vec{k}_\perp; \underline{p}) b_{js}^\dagger(1-y, -\vec{k}_\perp; \underline{p}) b_{i\sigma}(\underline{p}) \right. \\ & \left. + h_{ijl}^{\sigma s\lambda*}(y, \vec{k}_\perp) b_{i\sigma}^\dagger(\underline{p}) b_{js}(1-y, -\vec{k}_\perp; \underline{p}) a_{l\lambda}(y, \vec{k}_\perp; \underline{p}) \right\}. \end{aligned} \quad (\text{A21})$$

As discussed at the beginning of Sec. II, the multiple arguments of the creation and annihilation operators are defined by $a_{l\lambda}(y, \vec{k}_\perp; \underline{p}) \equiv a_{l\lambda}(yp^+, y\vec{p}_\perp + \vec{k}_\perp)$. The vertex functions

are

$$\begin{aligned}
h_{ijl}^{\pm\pm\pm}(y, \vec{k}_\perp) &= \mp e\sqrt{2\beta_i\beta_j\xi_l} \frac{k_\perp e^{\mp i\phi}}{(1-y)y^{3/2}}, \\
h_{ijl}^{\pm\pm\mp}(y, \vec{k}_\perp) &= \pm e\sqrt{2\beta_i\beta_j\xi_l} \frac{k_\perp e^{\pm i\phi}}{y^{3/2}}, \\
h_{ijl}^{\mp\pm\pm}(y, \vec{k}_\perp) &= 0, \\
h_{ijl}^{\mp\pm\mp}(y, \vec{k}_\perp) &= e\sqrt{2\beta_i\beta_j\xi_l} \frac{m_i(1-y) - m_j}{(1-y)\sqrt{y}}, \\
h_{ijl}^{\pm\pm 3}(y, \vec{k}_\perp) &= e\sqrt{\beta_i\beta_j\xi_l} \frac{m_i m_j y^2 - \mu_l^2(1-y) + k_\perp^2}{\mu_l(1-y)y^{3/2}}, \\
h_{ijl}^{\pm\pm 0}(y, \vec{k}_\perp) &= e\sqrt{\beta_i\beta_j\xi_l} \frac{m_i m_j y^2 + \tilde{\mu}_l^2(1-y) + k_\perp^2}{\mu_l(1-y)y^{3/2}}, \\
h_{ijl}^{\mp\pm 3}(y, \vec{k}_\perp) &= h_{ijl}^{\mp\pm 0}(y, \vec{k}_\perp) = \pm e\sqrt{\beta_i\beta_j\xi_l} \frac{(m_j - m_i)k_\perp e^{\mp i\phi}}{\mu_l(1-y)\sqrt{y}}.
\end{aligned} \tag{A22}$$

The gauge condition $\partial \cdot A_l = 0$ for the l th flavor is implemented as a projection onto a physical subspace. Only the polarization with opposite metric, $\lambda = 0$, contributes to $\partial \cdot A_l$; we have

$$\partial \cdot A_l = -i \frac{\mu_l}{\sqrt{\zeta}} \int \frac{d\vec{k}}{\sqrt{16\pi^3 k^+}} \left[a_{l0}(\vec{k}) e^{-i\vec{k} \cdot x} - a_{l0}^\dagger(\vec{k}) e^{i\vec{k} \cdot x} \right]. \tag{A23}$$

We require that physical states $|\psi_{\text{phys}}\rangle$ satisfy

$$\langle \psi_{\text{phys}} | \partial \cdot A_l | \psi_{\text{phys}} \rangle = 0, \tag{A24}$$

which is guaranteed if $|\psi_{\text{phys}}\rangle$ is annihilated by the positive-frequency part of $\partial \cdot A_l$ or, equivalently,

$$\mu_l a_{l0}(\vec{k}) |\psi_{\text{phys}}\rangle = 0. \tag{A25}$$

However, in the massless limit, the polarization vectors $e^{(3)}$ and $e^{(0)}$ become identical. This suggests that implementation of the gauge projection for a massless photon should involve the removal of not just the $e^{(0)}$ polarization but also the $e^{(3)}$ polarization, as one would expect on physical grounds.

To make this more precise, define two new polarizations

$$\tilde{e}^{(0,3)}(\vec{k}) = \frac{\mu_l}{2k^+} (e^{(3)} + e^{(0)}) \pm \frac{k^+}{2\mu_l} (e^{(0)} - e^{(3)}) \tag{A26}$$

before taking the massless limit. With use of the definitions in Eqs. (A13) and (A14), they can be written explicitly as

$$\tilde{e}^{(0,3)}(\vec{k}) = \frac{1}{k^+} \left(\frac{k_\perp^2 + \mu_l^2(1-\zeta)/2\zeta \pm (k^+)^2(1+\zeta)/2\zeta}{k^+}, \vec{k} \right). \tag{A27}$$

These have the useful properties of not being the same in the massless limit and of reducing, in Feynman gauge ($\zeta = 1$), to the standard choice of [8]

$$\tilde{e}^{(3)} = \frac{k - (n \cdot k)n}{n \cdot k} \text{ and } \tilde{e}^{(0)} = n, \tag{A28}$$

where n is the timelike four-vector that reduces to $n = (1, 0, 0, 0)$ in the frame where $\vec{k}_\perp = 0$. By requiring that

$$\sum_{\lambda=0,3} \tilde{e}^{(\lambda)}(\underline{k}) \tilde{a}_{l\lambda}(\underline{k}) = \sum_{\lambda=0,3} e^{(\lambda)}(\underline{k}) a_{l\lambda}(\underline{k}), \quad (\text{A29})$$

we find that

$$a_{l0}(\underline{k}) = \left(\frac{\mu_l}{2k^+} - \frac{k^+}{2\mu_l} \right) \tilde{a}_{l3} + \left(\frac{\mu_l}{2k^+} + \frac{k^+}{2\mu_l} \right) \tilde{a}_{l0} \quad (\text{A30})$$

and that the $\tilde{a}_{l\lambda}$ satisfy the same commutation relations as do the $a_{l\lambda}$.

The gauge projection (A25) now becomes, in the massless limit,

$$(\tilde{a}_{l0}(\underline{k}) - \tilde{a}_{l3}(\underline{k}))|\psi_{\text{phys}}\rangle = 0. \quad (\text{A31})$$

This removes from $|\psi_{\text{phys}}\rangle$ one null combination of \tilde{a}_{l0} and \tilde{a}_{l3} . The other null combination, $\tilde{a}_{l0} + \tilde{a}_{l3}$ remains but makes no contribution to physical quantities simply because it is null; instead, it represents the remaining gauge freedom not fixed by the Lorentz gauge condition [8]. For practical calculations of observables, it too can be removed, which makes the gauge projection equivalent to the removal of both the $e^{(0)}$ and $e^{(3)}$ polarizations.

For the massive PV photons, we apply the same projection. This could not be done for a physical massive vector particle, for which three polarizations must be retained. For the PV photons, which are regulators and not physical, the projection becomes part of the regularization prescription. In any case, in the infinite-mass limit, all such contributions from the PV photons disappear; removing them prior to taking this limit will not change the result.

Appendix B: Derivation of the Effective Hamiltonian

The effective Hamiltonian $\overline{\mathcal{P}}^-$ is constructed with use of the Baker–Hausdorff expansion (2.9) as applied to \mathcal{P}^- in (A18) and T in (3.3). The first commutator with the first term in \mathcal{P}^- is

$$[\mathcal{P}_{0a}^-, T] = \sum_{ijl\sigma s\lambda} \int dy d\vec{k}_\perp \int \frac{d\underline{p}}{\sqrt{16\pi^3 p^+}} \left[\frac{\mu_{l\lambda}^2 + (y\vec{p}_\perp + \vec{k}_\perp)^2}{y} \right] \quad (\text{B1})$$

$$\times t_{ijl}^{\sigma a\lambda}(y, \vec{k}_\perp) a_{l\lambda}^\dagger(y, \vec{k}_\perp; \underline{p}) b_{js}^\dagger(1-y, -\vec{k}_\perp; \underline{p}) b_{i\sigma}(\underline{p}). \quad (\text{B2})$$

For the second term, we have

$$[\mathcal{P}_{0b}^-, T] = \sum_{ijl\sigma s\lambda} \int dy d\vec{k}_\perp \int \frac{d\underline{p}}{\sqrt{16\pi^3 p^+}} \left[\frac{m_j^2 + ((1-y)\vec{p}_\perp - \vec{k}_\perp)^2}{1-y} - (m_i^2 + p_\perp^2) \right] \quad (\text{B3})$$

$$\times t_{ijl}^{\sigma a\lambda}(y, \vec{k}_\perp) a_{l\lambda}^\dagger(y, \vec{k}_\perp; \underline{p}) b_{js}^\dagger(1-y, -\vec{k}_\perp; \underline{p}) b_{i\sigma}(\underline{p}).$$

Due to cancellations between terms involving the transverse momentum \vec{p}_\perp , the combination of the two simplifies to

$$[\mathcal{P}_{0a}^- + \mathcal{P}_{0b}^-, T] = \sum_{ijl\sigma s\lambda} \int dy d\vec{k}_\perp \int \frac{d\underline{p}}{\sqrt{16\pi^3 p^+}} \left[\frac{m_j^2 + \vec{k}_\perp^2}{1-y} + \frac{\mu_{l\lambda}^2 + \vec{k}_\perp^2}{y} - m_i^2 \right] \quad (\text{B4})$$

$$\times t_{ijl}^{\sigma a\lambda}(y, \vec{k}_\perp) a_{l\lambda}^\dagger(y, \vec{k}_\perp; \underline{p}) b_{js}^\dagger(1-y, -\vec{k}_\perp; \underline{p}) b_{i\sigma}(\underline{p}).$$

A graphical representation is given in Fig. 1. Since this commutator already involves a net increase by one particle, no additional commutators with T need to be considered for these terms, in the chosen truncation. Similarly, the photon emission term in $\mathcal{P}_{\text{int}}^-$ already increases the particle number and need not be considered at all in any commutator. For the photon absorption term, however, two commutators must be considered.

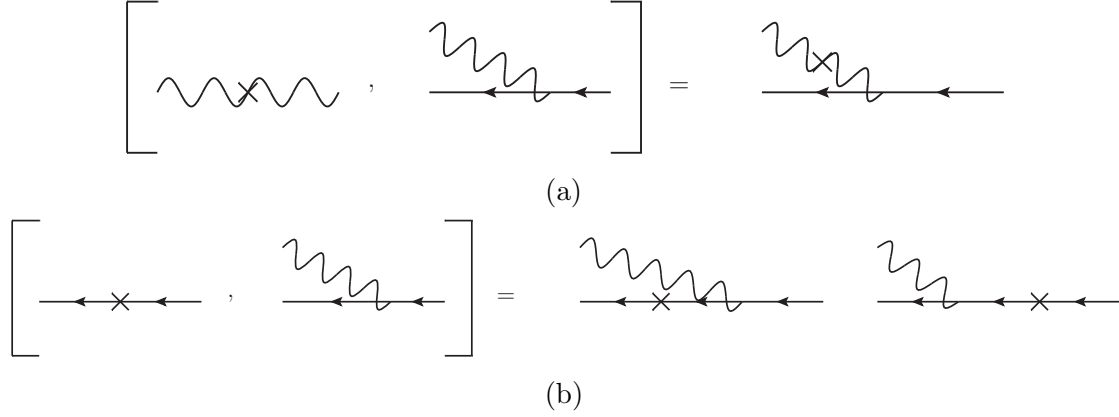


FIG. 1. Graphical representations of the terms in (a) $[\mathcal{P}_{0a}^-, T]$ and (b) $[\mathcal{P}_{0b}^-, T]$. Each diagram represents an operator that annihilates particles on the right and creates particles on the left. The crosses represent kinetic-energy contributions.

The first commutator with $\mathcal{P}_{\text{int}}^-$ generates terms that do not change particle number. They can be represented graphically as in Fig. 2. The commutator can be written as

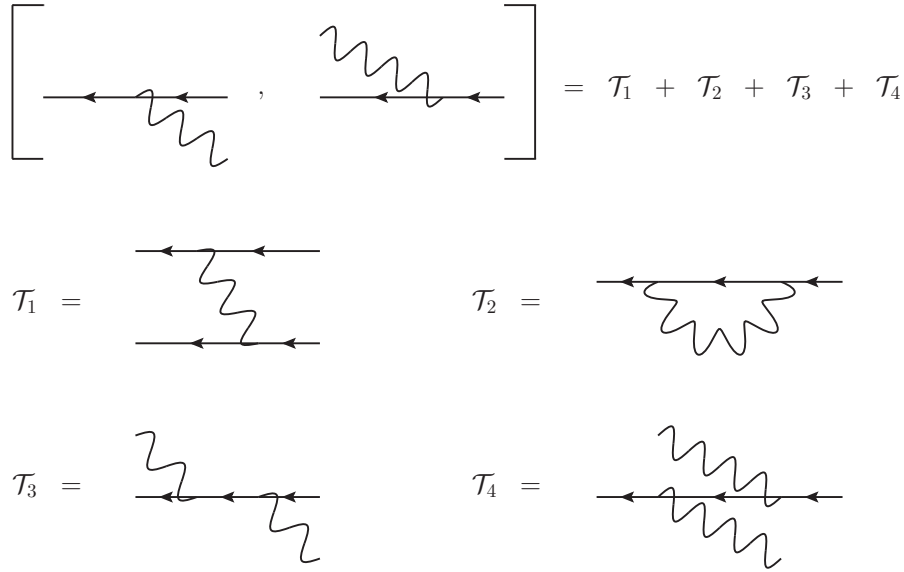


FIG. 2. Graphical representation of the terms in $[\mathcal{P}_{\text{int}}^-, T]$.

$$[\mathcal{P}_{\text{int}}^-, T] = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4, \quad (\text{B5})$$

with

$$\mathcal{T}_1 = - \sum_{ijl\sigma s\lambda} (-1)^l \epsilon^\lambda \sum_{i'j'\sigma's'} \int dy d\vec{k}_\perp \int \frac{d\underline{p}}{\sqrt{16\pi^3 p^+}} \int dy' d\vec{k}'_\perp \int \frac{d\underline{p}'}{\sqrt{16\pi^3}} \sqrt{p'^+} \quad (\text{B6})$$

$$\times h_{ijl}^{\sigma s \lambda *} (y, \vec{k}_\perp) t_{i'j'l}^{\sigma' s' \lambda} (y', \vec{k}'_\perp) \delta(y p^+ - y' p'^+) \delta(y \vec{p}_\perp + \vec{k}_\perp - y' \vec{p}'_\perp - \vec{k}'_\perp)$$

$$\times b_{j's'}^\dagger (1 - y', -\vec{k}'_\perp; \underline{p}') b_{i\sigma}^\dagger (\underline{p}) b_{js} (1 - y, -\vec{k}_\perp; \underline{p}) b_{i'\sigma'} (\underline{p}'),$$

$$\mathcal{T}_2 = \sum_{ij\sigma\sigma'} (-1)^i I_{ji}^{\sigma\sigma'} \int \frac{d\underline{p}}{p^+} b_{j\sigma}^\dagger (\underline{p}) b_{i\sigma'} (\underline{p}), \quad (\text{B7})$$

$$\mathcal{T}_3 = - \sum_{ijl\sigma s\lambda} (-1)^i \sum_{j'l's'\lambda'} \int dy d\vec{k}_\perp \int dy' d\vec{k}'_\perp \int \frac{d\underline{p}}{16\pi^3} h_{ijl}^{\sigma s \lambda *} (y, \vec{k}_\perp) t_{ij'l'}^{\sigma s' \lambda'} (y', \vec{k}'_\perp) \quad (\text{B8})$$

$$\times a_{l'\lambda'}^\dagger (y', \vec{k}'_\perp; \underline{p}) b_{j's'}^\dagger (1 - y', -\vec{k}'_\perp; \underline{p}) b_{js} (1 - y, -\vec{k}_\perp; \underline{p}) a_{l\lambda} (y, \vec{k}_\perp; \underline{p}),$$

$$\mathcal{T}_4 = \sum_{ijl\sigma s\lambda} (-1)^j \int dy d\vec{k}_\perp \int \frac{d\underline{p}}{\sqrt{16\pi^3 p^+}} \sum_{i'l'\sigma'\lambda'} \int dy' d\vec{k}'_\perp \int \frac{d\underline{p}'}{\sqrt{16\pi^3}} \sqrt{p'^+} \quad (\text{B9})$$

$$\times \delta((1 - y)p^+ - (1 - y')p'^+) \delta((1 - y)\vec{p}_\perp - \vec{k}_\perp - (1 - y')\vec{p}'_\perp + \vec{k}'_\perp)$$

$$\times h_{ijl}^{\sigma s \lambda *} (y, \vec{k}_\perp) t_{i'l'}^{\sigma' s' \lambda'} (y', \vec{k}'_\perp) a_{l'\lambda'}^\dagger (y', \vec{k}'_\perp; \underline{p}') b_{i\sigma}^\dagger (\underline{p}) b_{i'\sigma'} (\underline{p}') a_{l\lambda} (y, \vec{k}_\perp; \underline{p}),$$

and

$$I_{ji}^{\sigma\sigma'} = (-1)^i \sum_{i'ls\lambda} (-1)^{i'+l} \epsilon^\lambda \int \frac{dy d\vec{k}_\perp}{16\pi^3} h_{ji'l}^{\sigma s \lambda *} (y, \vec{k}_\perp) t_{ii'l}^{\sigma' s \lambda} (y, \vec{k}_\perp). \quad (\text{B10})$$

The first term, \mathcal{T}_1 , does not contribute to the chosen truncation to one fermion. The spin dependence of the self-energy term \mathcal{T}_2 can be simplified upon the observation that the dependence of $t_{ii'l}^{\sigma' s \lambda} (y, \vec{k}_\perp)$ on the azimuthal angle of \vec{k}_\perp must follow the pattern of dependence in $h_{ii'l}^{\sigma' s \lambda} (y, \vec{k}_\perp)$, in order that T conserve J_z . The azimuthal integral in $I_{ji}^{\sigma\sigma'}$ then implies that

$$I_{ji}^{\sigma\sigma'} = \delta_{\sigma\sigma'} I_{ji}, \quad (\text{B11})$$

with I_{ji} real and independent of the spin projection σ . This then simplifies \mathcal{T}_2 to be

$$\mathcal{T}_2 = \sum_{ijs} (-1)^i \int \frac{d\underline{p}}{p^+} I_{ji} b_{js}^\dagger (\underline{p}) b_{is} (\underline{p}). \quad (\text{B12})$$

The second commutator $[[\mathcal{P}_{\text{int}}^-, T], T]$ generates several more terms, all of which increase particle number by one; however, most of these do not contribute. Graphical representations are given in Fig. 3. In particular, $[\mathcal{T}_1, T]$ contributes nothing for our truncation. The next possibility $[\mathcal{T}_2, T]$ does contribute fully and, of course, has a structure very similar to that of $[\mathcal{P}_{0b}^-, T]$:

$$[\mathcal{T}_2, T] = \sum_{ijl\sigma s\lambda} \int dy d\vec{k}_\perp \int \frac{d\underline{p}}{\sqrt{16\pi^3 p^+}} \left[\sum_{i'} \frac{I_{ji'}}{1 - y} t_{ii'l}^{\sigma s \lambda} (y, \vec{k}_\perp) - \sum_{j'} (-1)^{i+j'} t_{j'jl}^{\sigma s \lambda} (y, \vec{k}_\perp) I_{j'i} \right] \quad (\text{B13})$$

$$\times a_{l\lambda}^\dagger (y, \vec{k}_\perp; \underline{p}) b_{js}^\dagger (1 - y, -\vec{k}_\perp; \underline{p}) b_{i\sigma} (\underline{p}).$$

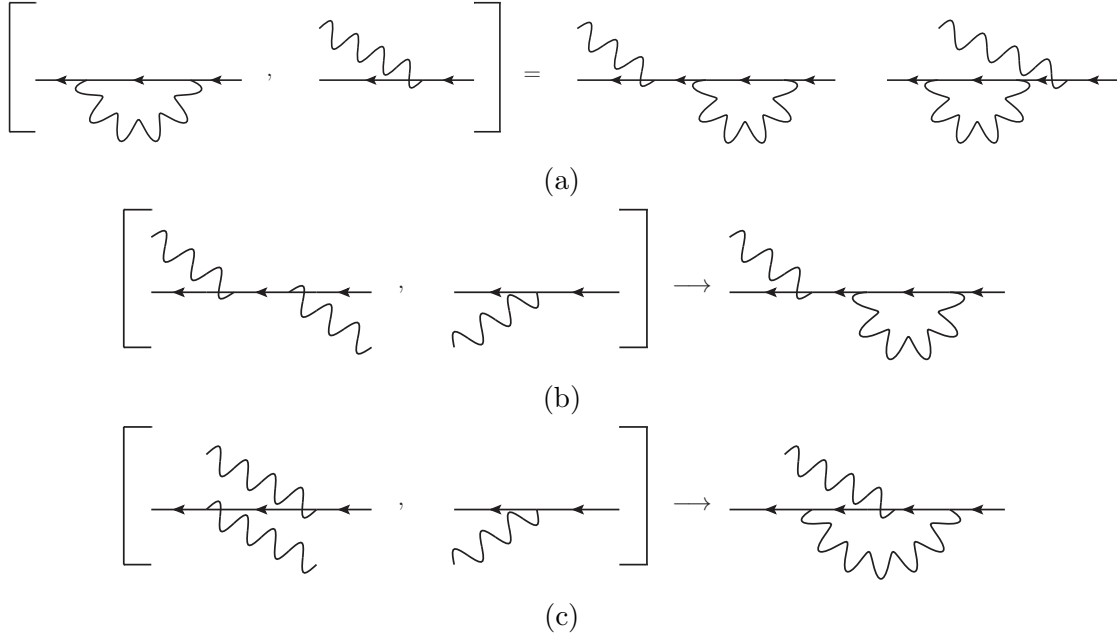


FIG. 3. Graphical representation of the contributions from $[[\mathcal{P}_{\text{int}}^-, T], T]$, with (a), (b), and (c) corresponding to $[\mathcal{T}_2, T]$, $[\mathcal{T}_3, T]$, and $[\mathcal{T}_4, T]$, respectively. $[\mathcal{T}_1, T]$ does not contribute at all for the chosen truncation.

The third term contains several pieces but contributes only

$$[\mathcal{T}_3, T] \rightarrow - \sum_{ijl\sigma s\lambda} \int dy d\vec{k}_\perp \int \frac{d\underline{p}}{\sqrt{16\pi^3 p^+}} \sum_{j'} (-1)^{i+j'} t_{j'jl}^{\sigma s\lambda}(y, \vec{k}_\perp) I_{j'i} \quad (\text{B14})$$

$$\times a_{i\lambda}^\dagger(y, \vec{k}_\perp; \underline{p}) b_{js}^\dagger(1-y, -\vec{k}_\perp; \underline{p}) b_{i\sigma}(\underline{p}),$$

which combines with the second term of $[\mathcal{T}_2, T]$. The fourth also contains several pieces; all that remains in the given truncation is

$$[\mathcal{T}_4, T] \rightarrow \sum_{ijl\sigma s\lambda} \int dy d\vec{k}_\perp \int \frac{d\underline{p}}{\sqrt{16\pi^3 p^+}} V_{ijl}^{\sigma s\lambda}(y, \vec{k}_\perp) a_{i\lambda}^\dagger(y, \vec{k}_\perp; \underline{p}) b_{js}^\dagger(1-y, -\vec{k}_\perp; \underline{p}) b_{i\sigma}(\underline{p}), \quad (\text{B15})$$

where

$$V_{ijl}^{\sigma s\lambda}(y, \vec{k}_\perp) = \sum_{i'j'l'\sigma'\lambda'} (-1)^{i'+j'+l'} \epsilon^{\lambda'} \int \frac{dy' d\vec{k}'_\perp}{16\pi^3} \frac{\theta(1-y-y')}{(1-y')^{1/2}(1-y)^{3/2}} \quad (\text{B16})$$

$$\times h_{jj'l'}^{ss'\lambda'*}(\frac{y'}{1-y}, \vec{k}'_\perp + \frac{y'}{1-y} \vec{k}_\perp) t_{i'j'l}^{\sigma'\lambda}(y, \vec{k}_\perp + \frac{y}{1-y'} \vec{k}'_\perp) t_{ii'l'}^{\sigma\sigma'\lambda'}(y', \vec{k}'_\perp).$$

This is a vertex correction, as illustrated in Fig. 3(c).

The effective Hamiltonian for the given truncation is then the sum of all these terms, according to the Baker–Hausdorff expansion. A graphical representation is shown in Fig. 4.

The final expression is

$$\begin{aligned}
\overline{\mathcal{P}}^- = & \sum_{ijs} (-1)^i \int d\underline{p} \left[\delta_{ij} \frac{m_i^2 + p_\perp^2}{p^+} + \frac{I_{ji}}{p^+} \right] b_{js}^\dagger(\underline{p}) b_{is}(\underline{p}) \\
& + \sum_{l\lambda} (-1)^l \epsilon^\lambda \int d\underline{p} \frac{\mu_{l\lambda}^2 + p_\perp^2}{p^+} a_{l\lambda}^\dagger(\underline{p}) a_{l\lambda}(\underline{p}) \\
& + \sum_{ijls\sigma\lambda} \int dy d\vec{k}_\perp \int \frac{d\underline{p}}{\sqrt{16\pi^3 p^+}} \left\{ h_{ijl}^{\sigma s\lambda}(y, \vec{k}_\perp) + \frac{1}{2} V_{ijl}^{\sigma s\lambda}(y, \vec{k}_\perp) \right. \\
& \quad + \left[\frac{m_j^2 + k_\perp^2}{1-y} + \frac{\mu_{l\lambda}^2 + k_\perp^2}{y} - m_i^2 \right] t_{ijl}^{\sigma s\lambda}(y, \vec{k}_\perp) \\
& \quad + \frac{1}{2} \sum_{i'} \frac{I_{ji'}}{1-y} t_{ii'l}^{\sigma s\lambda}(y, \vec{k}_\perp) - \sum_{j'} (-1)^{i+j'} t_{j'jl}^{\sigma s\lambda}(y, \vec{k}_\perp) I_{j'i} \left. \right\} \\
& \quad \times a_{l\lambda}^\dagger(y, \vec{k}_\perp; \underline{p}) b_{js}^\dagger(1-y, -\vec{k}_\perp; \underline{p}) b_{i\sigma}(\underline{p}) \\
& + \sum_{ijls\sigma\lambda} \int dy d\vec{k}_\perp \int \frac{d\underline{p}}{\sqrt{16\pi^3 p^+}} h_{ijl}^{\sigma s\lambda*}(y, \vec{k}_\perp) b_{i\sigma}^\dagger(\underline{p}) b_{js}(1-y, -\vec{k}_\perp; \underline{p}) a_{l\lambda}(y, \vec{k}_\perp; \underline{p}) \\
& + \sum_{ijl\sigma s\lambda} (-1)^j \sum_{i'l'\sigma'\lambda'} \int dy d\vec{k}_\perp \int \frac{d\underline{p}}{\sqrt{16\pi^3 p^+}} \int dy' d\vec{k}'_\perp \int \frac{d\underline{p}'}{\sqrt{16\pi^3}} \sqrt{p'^+} \\
& \quad \times \delta((1-y)p^+ - (1-y')p'^+) \delta((1-y)\vec{p}_\perp - \vec{k}_\perp - (1-y')\vec{p}'_\perp + \vec{k}'_\perp) \\
& \quad \times h_{ijl}^{\sigma s\lambda*}(y, \vec{k}_\perp) t_{i'jl'}^{\sigma' s\lambda'}(y', \vec{k}'_\perp) a_{l'\lambda'}^\dagger(y', \vec{k}'_\perp; \underline{p}') b_{i\sigma}^\dagger(\underline{p}) b_{i'\sigma'}(\underline{p}') a_{l\lambda}(y, \vec{k}_\perp; \underline{p}) \\
& - \sum_{ijl\sigma s\lambda} (-1)^i \sum_{j'l's'\lambda'} \int dy d\vec{k}_\perp \int \frac{d\underline{p}}{16\pi^3} \int dy' d\vec{k}'_\perp h_{ijl}^{\sigma s\lambda*}(y, \vec{k}_\perp) t_{ij'l'}^{\sigma s\lambda'}(y', \vec{k}'_\perp) \\
& \quad \times a_{l'\lambda'}^\dagger(y', \vec{k}'_\perp; \underline{p}') b_{j's'}^\dagger(1-y', -\vec{k}'_\perp; \underline{p}') b_{js}(1-y, -\vec{k}_\perp; \underline{p}) a_{l\lambda}(y, \vec{k}_\perp; \underline{p})
\end{aligned} \tag{B17}$$

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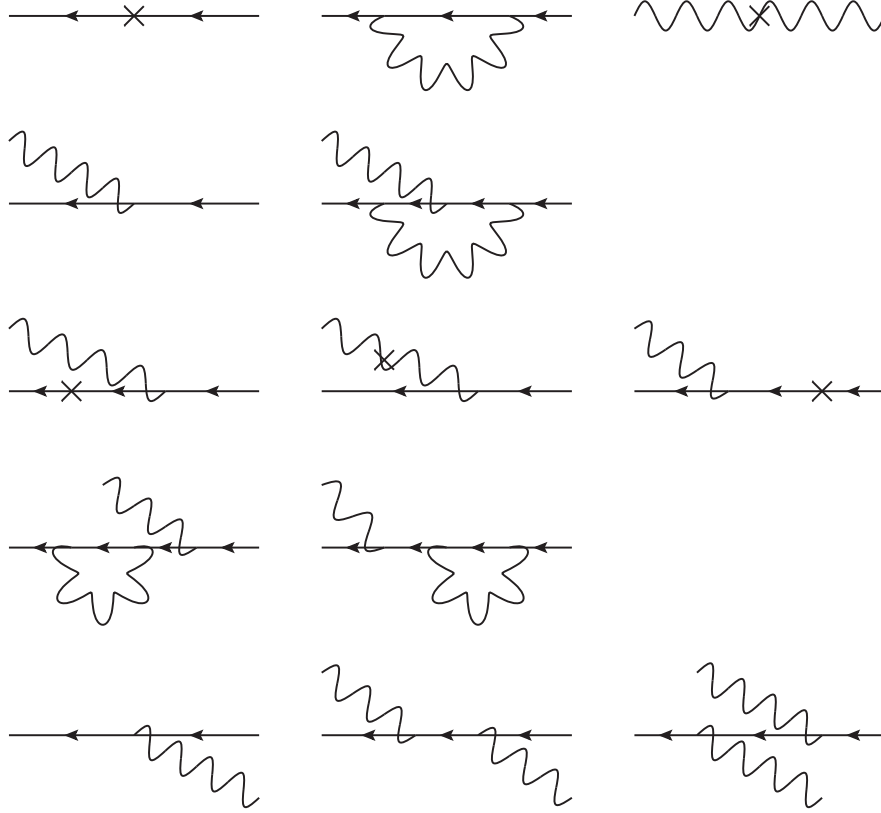


FIG. 4. Graphical representation of the effective Hamiltonian $\overline{\mathcal{P}}^-$, limited to those terms that contribute in the given truncation of the operator T .

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